A REMARKABLE 20-CROSSING TANGLE

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Abstract. For any positive integer r, we exhibit a prime knot K_r with $(20 \cdot 2^{r-1} + 1)$ crossings whose Jones polynomial $V(K_r)$ is equal to 1 modulo 2^r. Our construction rests on a certain 20-crossing prime tangle T_{20} which is undetectable by the Kauffman bracket polynomial pair mod 2.

1. Introduction

In [7], M. B. Thistlethwaite gave two 2–component links and one 3–component link which are nontrivial and yet have the same Jones polynomial as the corresponding unlink U^2 and U^3, respectively. These were the first known examples of nontrivial links undetectable by the Jones polynomial. Shortly thereafter, it was shown in [2] that, for any integer $k \geq 2$, there exist infinitely many nontrivial k–component links whose Jones polynomial is equal to that of the k–component unlink U^k. Yet the corresponding problem for 1–component links, i.e. for knots, is widely open: *does there exist a nontrivial knot K whose Jones polynomial is equal to that of the unknot U^1, namely to 1?*

We shall consider here the following weaker problem, consisting in looking for nontrivial knots K whose Jones polynomial is *congruent modulo some integer* to that of the unknot U^1.

Problem 1.1. *Given any integer $m \geq 2$, does there exist a prime knot K whose Jones polynomial $V(K)$ satisfies $V(K) \equiv 1 \mod m$?*

Naturally, for any two Laurent polynomials f, g in $\mathbb{Z}[t, t^{-1}]$, the notation $f \equiv g \mod m$ means that there exists an element $h \in \mathbb{Z}[t, t^{-1}]$ such that $f - g = m \cdot h$. This is equivalent to require that, for each $i \in \mathbb{Z}$, the coefficients α_i and β_i of t^i in f and g, respectively, are congruent modulo m as integers.

A result of M. B. Thistlethwaite [6] states that, for an alternating knot K with n crossings, the span of $V(K)$ is exactly n and the coefficients of the terms of maximal and minimal degree in V_K are both ± 1. In particular, for any $m \geq 2$, *there is no alternating knot with trivial Jones polynomial modulo m.*

Using the *Mathematica* package `KnotTheory` of the KnotAtlas project [1], it is easy to find knots which are solutions of Problem 1.1 for the moduli $m = 2, 3$ and 4. For $m = 5$ there is no solution of Problem 1.1 among the knots up to 16 crossings. The following table gives the number of solutions of Problem 1.1 for the moduli 2

2010 *Mathematics Subject Classification.* 57M27.

Key words and phrases. Jones polynomial, Kauffman bracket, algebraic tangle.

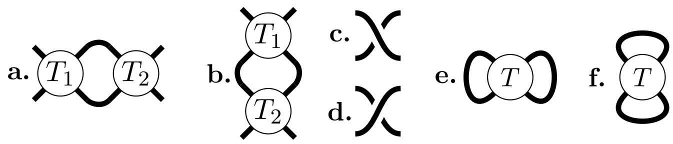


FIGURE 1. **a.** Tangle $T_1 + T_2$. **b.** Tangle $T_1 * T_2$. **c.** Tangle 1. **d.** Tangle -1. **e.** Closure $\mathrm{den}(T)$. **f.** Closure $\mathrm{num}(T)$.

to 5 up to 16 crossings, respectively:

m	≤ 11	12	13	14	15	16
2	0	4	9	35	140	582
3	0	1	0	1	2	26
4	0	0	0	0	1	0
5	0	0	0	0	0	0

In this note, we shall solve Problem 1.1 for all moduli m which are powers of 2. That is, given any integer $r \geq 1$, we shall construct infinitely many prime knots whose Jones polynomial is trivial mod $m = 2^r$. Our construction rests on a certain 20-crossing prime tangle T_{20} whose Kauffman bracket polynomial pair is trivial mod 2.

The paper is structured as follows. Section 2 is devoted to basic tangle constructions. In Section 3 we describe our tangle T_{20}, we prove that it is prime and we construct a family of prime knots K_r from a prime tangle M_r which is composed of 2^r copies of the tangle T_{20}. In Section 4, we compute the Kauffman bracket pair of the tangle M_r modulo 2^r. In Section 5, we prove that the knots K_r for $r \geq 1$ are distinct and that the Jones polynomial of K_r is trivial modulo 2^r. The paper ends with some concluding remarks.

2. Basic tangle constructions

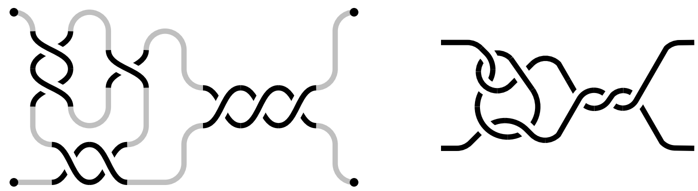
We shall use the same notation as in [2]. In particular, if T_1, T_2 are two tangles with 4 boundary points, we denote by $T_1 + T_2$ their *horizontal sum* and by $T_1 * T_2$ their *vertical sum* (see **a** and **b** of Figure 1). The tangle 1 denotes a single crossing as in **c** of Figure 1, while the tangle -1 denotes its opposite version as in **d** of Figure 1. More generally, if T is a tangle, then $-T$ denotes the tangle obtained from T by switching the signs of all crossings in T.

As usual, for $k \in \mathbb{N} \setminus \{0\}$ we define the tangles

$$\begin{aligned} k &= 1 + \ldots + 1, & -k &= (-1) + \ldots + (-1), \\ 1/k &= 1 * \ldots * 1, & -1/k &= (-1) * \ldots * (-1), \end{aligned}$$

with k terms in each expression. These are particular cases of *algebraic tangles*, namely tangles constructed recursively from the tangles ± 1 using horizontal and vertical sums.

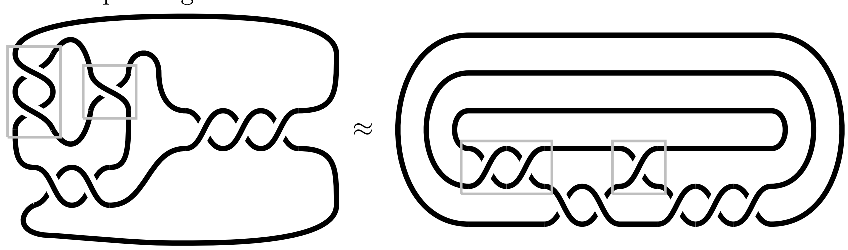
Example 2.1. Let $T_{8,10}$ be the tangle $(((1/2) + 1) * 2) + (-3)$. (See next example for the choice of this name.) Diagrams corresponding to $T_{8,10}$ are



The parts in black correspond to the tangles $1/2$, 1, 2 and -3, respectively, while the gray strands depict the connections between them. The four marked points represent the extremities of the global tangle. The rightmost diagram is a "smoother" representation of the tangle $T_{8,10}$.

If T is a tangle, we denote by $\mathrm{den}(T)$ and $\mathrm{num}(T)$ the link diagrams obtained by gluing the extremities of T as in **e.** and **f.** of Figure 1, respectively.

Example 2.2. Consider again the tangle $T_{8,10}$ of Example 2.1. Here are $\mathrm{num}(T_{8,10})$ and an isotopic diagram.



The isotopy is obtained from the left diagram by rotating counterclockwise the left block in gray and clockwise the right one. Looking at Dale Rolfen's knot table [5], we remark that $\mathrm{num}(T_{8,10})$ is a diagram of $K_{8,10}$ which is the 10^{th} prime knot with 8 crossings.

3. Lickorish theory of prime tangles and links

For a more geometric point of view on tangles, we need the following refinement as in [2]. A *geometric tangle* is a pair (B,t), where B is a 3-ball and t is a proper 1-submanifold of B meeting the boundary of B in four points.

To each tangle T we associate in a natural way a geometric tangle (B,t) where B is a Euclidean 3-ball whose boundary meets the projection plane P in an "equatorial" circle circumscribing the tangle T, and where t is obtained from T drawn in P by making small vertical perturbations near crossings.

A geometric tangle (B,t) is *untangled* if it is isotopic to a pair (B,t_0), where t_0 is the union of two parallel line segments in the projection plane.

Definition 3.1. A tangle (B,t) is *prime* if it has the following properties:
i) Any 2-sphere in B, which meets t transversely in two points, bounds in B a ball meeting t in an unknotted spanning arc.
ii) (B,t) is not untangled.

A tangle T is said to be *prime* if its associated geometric tangle (B,t) is. Prime tangles supply an easy-to-use machinery to construct prime knots or links as shown by the following result due to Lickorish [3].

Theorem 3.2. *If T and U are prime tangles, then* $\mathrm{den}(T*U)$ *is a prime link.*

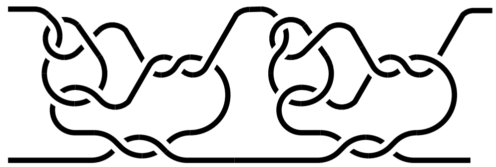


FIGURE 2. The key tangle $M_1 = T_{20}$.

Following one of the methods proposed in [3] to prove that a tangle is prime (more precisely, that used for the tangle in Figure **2.a** of [3]) we introduce the notion of a strongly prime tangle.

Definition 3.3. A tangle T is said to be *strongly prime* if it is not untangled and if $T +)($ is isotopic to $)($.

A direct consequence of the definition is that *if $T_1, ..., T_n$ are strongly prime tangles such that $T = T_1 + \ldots + T_n$ is not untangled, then T is also strongly prime.* As the terminology suggests, a strongly prime tangle is a particular prime tangle.

Proposition 3.4. *Every strongly prime tangle is a prime tangle.*

Proof. Let T be a strongly prime tangle and (B, t) the corresponding geometric tangle. By hypothesis, T is not untangled and so is (B, t). Assume there exists a ball in (B, t) meeting the arcs of t in just one knotted arc. Then such a knotted arc-ball pair would persist as a summand on any knot created by adding another tangle to T. However, as T is strongly prime we have $\mathrm{num}(T +)() \approx \mathrm{num}()() \approx \bigcirc$ and so (B, t) certainly cannot contain a summand. $\square$

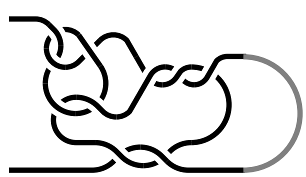
3.1. **The tangles M_r.** We now introduce a family of tangles which includes the tangle T_{20}, the cornerstone of our solution to Problem 1.1 for moduli m which are powers of 2.

Definition 3.5. We define the tangles T_{10}, T_{20} and M_r for $r \geq 1$ as follows:
– *i*) $T_{10} = T_{8,10} * 2 = ((((1/2) + 1) * 2) + (-3)) * 2$;
– *ii*) $T_{20} = T_{10} + (-T_{10})$;
– *iii*) $M_1 = T_{20}$ and $M_r = M_{r-1} + M_{r-1}$ for $r \geq 2$.

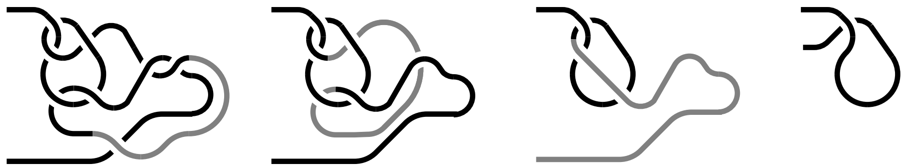
Tangle M_1 is depicted in Figure 2.

Proposition 3.6. *For all $r \geq 1$, the tangle M_r is prime.*

Proof. Let us first show that the tangle T_{10} is strongly prime. Linking the NE and SE extremities of T_{10} we obtain the diagram below:



From the following sequence of deformations, where the isotoped piece is in gray,

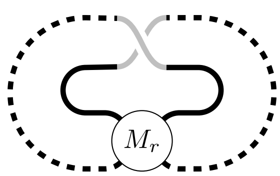


we establish that $T_{10} +)($ is isotopic to $)($. The arc joining the north extremities of T_{10} cannot be unknotted. Indeed, if we glue its two extremities we obtain, by Example 2.1, the knot $K_{8,10}$ which is certainly different from the unknot. Therefore T_{10} is strongly prime. As the tangle $-T_{10}$ is obtained from T_{10} by switching the signs of the crossings, the tangle $-T_{10}$ is also strongly prime. Since T_{10} and $-T_{10}$ are strongly prime tangles, we have $T_{20} +)(= T_{10} + (-T_{10}) +)($ which is isotopic to $T_{10} +)($ and hence to $)($. To prove that T_{20} is strongly prime, it remains to establish that it is not untangled. The tangle T_{20} is composed of two arcs, the first one going from NE to NW and the second one going from SE to SW. If we glue the two extremities of the first arc we obtain a knot K with two summands, namely $K_{8,10}$ and its mirror. In particular K is different from the unknot and so T_{20} is not untangled, implying that T_{20} is strongly prime. As M_r is the sum of 2^r copies of M_1, the tangle $M_r +)($ is isotopic to $)($. As for T_{20}, if we glue the arc of M_r going from NE to NW we obtain a knot K with 2^r summands, namely 2^{r-1} copies of $K_{8,10}$ and 2^{r-1} copies of the mirror of $K_{8,10}$. Hence the tangle M_r is also strongly prime. We conclude using Proposition 3.4. $\square$

3.2. **The knot K_r.**

Definition 3.7. For $r \in \mathbb{N} \setminus \{0\}$, we define K_r to be the knot represented by the diagram $\mathrm{den}(1 * M_r)$.

By definition, the tangle M_r has $2^{r-1} \times 20$ crossings. So the knot K_r has at most $1 + 2^{r-1} \times 20$ crossings. As 1 and M_r are prime tangles (the tangle 1 is obviously strongly prime), Theorem 3.2 guarantees that the link K_r is prime. We have already observed that the tangle M_r is the union of two arcs, the first one going from NW to NE and the second one from SW to SE. As illustrated by the diagram



the link diagram $\mathrm{den}(1 * M_r)$ has exactly one component: if we travel along the dotted arc we must meet the undotted one. The following proposition summarizes these remarks.

Proposition 3.8. *For each $r \geq 1$, the link K_r is a prime knot with at most $1 + 2^{r-1} \times 20$ crossings.*

An elegant mutant version of knot K_1 can be found on Figure 4. In Section 5 we will prove that the Jones polynomial of the prime knot K_r is trivial modulo 2^r.

4. The Kauffman bracket pair of a tangle

In this section we recall the definition of the Kauffman bracket pair of a tangle. This notion is very powerful to determine the Jones polynomial for a knot obtained from an algebraic tangle, which is the case for our knots K_r.

Let T be a tangle with 4 endpoints. The Kauffman bracket $\langle T\rangle$ of T is a linear combination of two formal symbols $\langle 0\rangle$ and $\langle\infty\rangle$ with coefficients in the ring of Laurent polynomials $\Lambda = \mathbb{Z}[t, t^{-1}]$. The bracket $\langle T\rangle$ may be computed with the usual rules of the Kauffman bracket polynomial, namely:

– $\langle \bigcirc \rangle = 1$;
– $\langle \bigcirc \amalg T\rangle = \delta\,\langle T\rangle$ where $\delta = -t^{-2} - t^{2}$;
– $\left\langle \times \right\rangle = t^{-1}\,\langle \asymp \rangle + t\,\langle)(\rangle$.

Thus, after removing the crossings and all free loops using the rules above, we end up with a unique expression of the form $\langle T\rangle = f(T)\,\langle 0\rangle + g(T)\,\langle\infty\rangle$ where 0 is the tangle $\asymp$ and ∞ is $)($. The notation 0 and ∞ comes from the shape of the link obtained when we take the den closure of the corresponding tangles.

We define the *bracket pair* $\mathrm{br}(T)$ of T as

$$\mathrm{br}(T) = \begin{bmatrix} f(T) \\ g(T) \end{bmatrix} \in \Lambda^2.$$

For example we compute

$$\langle -1\rangle = \left\langle \times \right\rangle = t^{-1}\,\langle \asymp \rangle + t\,\langle)(\rangle = t^{-1}\,\langle 0\rangle + t\,\langle\infty\rangle,$$

which implies $\mathrm{br}(-1) = \begin{bmatrix} t^{-1} \\ t \end{bmatrix}$. Since by definition of $-T$, we have $f(-T) = f(T)|_{t\leftarrow t^{-1}}$ and $g(-T) = g(T)|_{t\leftarrow t^{-1}}$ we obtain $\mathrm{br}(1) = \begin{bmatrix} t \\ t^{-1} \end{bmatrix}$.

Proposition 2.2 of [2] gives computation rules for the bracket pair of the horizontal and vertical sum of tangles together with the bracket of the num and den closures of tangles. Here they are.

Proposition 4.1. *For two tangles T and U, we have:*

– *i)* $\mathrm{br}(T+U) = \begin{bmatrix} f(T)f(U) \\ f(T)g(U) + g(T)f(U) + \delta g(T)g(U) \end{bmatrix}$;

– *ii)* $\mathrm{br}(T*U) = \begin{bmatrix} \delta f(T)f(U) + f(T)g(U) + g(T)f(U) \\ g(T)g(U) \end{bmatrix}$;

– *iii)* $\langle \mathrm{num}(T)\rangle = \delta f(T) + g(T)$ *and* $\langle \mathrm{den}(T)\rangle = f(T) + \delta g(T)$.

A direct computation from the expression of $\mathrm{br}(1)$ gives

$$\mathrm{br}(2) = \begin{bmatrix} t^2 \\ -t^{-4}+1 \end{bmatrix}, \quad \mathrm{br}(3) = \begin{bmatrix} t^3 \\ t^{-7} - t^{-3} + t \end{bmatrix} \quad \text{and} \quad \mathrm{br}(1/2) = \begin{bmatrix} 1 - t^4 \\ t^{-2} \end{bmatrix}.$$

Using these values, we determine the bracket pair of $T_{8,10} = (((1/2)+1)*2)+(-3)$:

$$\mathrm{br}(T_{8,10}) = \begin{bmatrix} -2\,t^{-6} + 2\,t^{-2} - 2\,t^2 + t^6 \\ -2\,t^{-4} + 3 - 4\,t^4 + 3\,t^8 - 2\,t^{12} + t^{16} \end{bmatrix}. \tag{1}$$

Notation. In the sequel, for a tangle T and an integer $m \geq 2$, we will denote by $\mathrm{br}_m(T)$ the bracket pair of T modulo m.

4.1. **The case of tangle T_{20}.** We now analyze the bracket pair of the tangle T_{20} introduced in Definition 3.5.

Lemma 4.2. *The bracket pair* $\mathrm{br}_2(T_{20})$ *is equal to* $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. *Moreover, the leading term of* $f(T_{20})$ *is* $2\,t^{28}$ *and that of* $g(T_{20})$ *is* $2\,t^{26}$.

Proof. By relation (1), we obtain

$$\mathrm{br}(T_{10}) = \mathrm{br}(T_{8,10} * 2) = \begin{bmatrix} 2\,t^{-10} - 2\,t^{-6} + 2\,t^{-2} - 2\,t^{6} + 2\,t^{10} - 2\,t^{14} + t^{18} \\ 2\,t^{-8} - 5\,t^{-4} + 7 - 7\,t^{4} + 5\,t^{8} - 3\,t^{12} + t^{16} \end{bmatrix}.$$

Replacing t by $-t$, we get

$$\mathrm{br}(-T_{10}) = \begin{bmatrix} t^{-18} - 2\,t^{-14} + 2\,t^{-10} - 2\,t^{-6} + 2\,t^{2} - 2\,t^{6} + 2\,t^{10} \\ t^{-16} - 3\,t^{-12} + 5\,t^{-8} - 7\,t^{-4} + 7 - 5\,t^{4} + 2\,t^{8} \end{bmatrix}.$$

The formula for the computation of $\mathrm{br}(T_{10} + (-T_{10}))$ given in *ii*) of Proposition 4.1 implies that the leading term of $f(T_{20})$ is $t^{18} \cdot 2\,t^{10} = 2\,t^{28}$ and that of $g(T_{20})$ is

$$t^{18} \cdot 2\,t^{8} + t^{16} \cdot 2\,t^{10} - t^{2} \cdot t^{16} \cdot 2\,t^{8} = 2\,t^{26}$$

as expected. Taking coefficient modulo 2, we obtain

$$\mathrm{br}_2(T_{10}) = \begin{bmatrix} t^{18} \\ t^{-4} + 1 + t^{4} + t^{8} + t^{12} + t^{16} \end{bmatrix},$$
$$\mathrm{br}_2(-T_{10}) = \begin{bmatrix} t^{-18} \\ t^{-16} + t^{-12} + t^{-8} + t^{-4} + 1 + t^{4} \end{bmatrix},$$

and so $\mathrm{br}_2(T_{20}) = \mathrm{br}_2(T_{10} + (-T_{10})) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. $\square$

4.2. **The general case of tangles M_r.**

Proposition 4.3. *For all* $r \geq 1$, *we have* $\mathrm{br}_{2^r}(M_r) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. *Moreover, the leading term of* $f(M_r)$ *is* $\ell_r = \left(2\,t^{28}\right)^{2^{r-1}}$ *while that of* $g(M_r)$ *is* $t^{-2}\ell_r$.

Proof. By induction on $r \geq 1$. The case $r = 1$ is Lemma 4.2. Assume now $r \geq 2$. By the induction hypothesis, we have $f(M_{r-1}) \equiv 1 \mod 2^{r-1}$ and $g(M_{r-1}) \equiv 0 \mod 2^{r-1}$. Hence, there exist two Laurent polynomials P and Q in $\mathbb{Z}[t^{-1}, t]$ such that the relations $f(M_{r-1}) = 1 + 2^{r-1}P$ and $g(M_{r-1}) = 2^{r-1}Q$ hold. By Proposition 4.1 and formula $\mathrm{br}(M_r) = \mathrm{br}(M_{r-1} + M_{r-1})$ we obtain:

$$\begin{aligned}\mathrm{br}(M_r) &= \begin{bmatrix} f(M_{r-1})^2 \\ 2g(M_{r-1})\,f(M_{r-1}) + \delta g(M_{r-1})^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 + 2^r P + 2^{2r-2}P^2 \\ 2^r Q + 2^{2r-1}PQ + \delta 2^{2r-2}Q^2 \end{bmatrix}.\end{aligned}$$

As $2r - 2 \geq r$ holds since $r \geq 2$, we find $\mathrm{br}_{2^r}(M_r) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. Let us now prove the statement about the leading terms. Since $f(M_r)$ is equal to $f(M_{r-1})^2$, the induction hypothesis implies that the leading term of $f(M_r)$ is the square of the leading term of $f(M_{r-1})$, *i.e.*, the square of ℓ_{r-1}. Since ℓ_{r-1}^2 is equal to ℓ_r, we have the desired result for the leading term of $f(M_r)$. Denoting by $\mathrm{lt}(P)$ the leading term of a Laurent polynomial $P \in \Lambda$, we have

$$\mathrm{lt}(g(M_r)) = \mathrm{lt}\left(2g(M_{r-1})f(M_{r-1}) + \delta\, g(M_{r-1})^2\right).$$

By the induction hypothesis, we compute

$$\begin{aligned}\mathrm{lt}(g(M_{r-1})\, f(M_{r-1})) &= t^{-2}\ell_{r-1}\cdot \ell_{r-1} = t^{-2}\ell_{r-1}^2 = t^{-2}\ell_r\\ \mathrm{lt}(\delta\, g(M_{r-1}^2) &= -t^2\cdot (t^{-2}\ell_{r-1})^2 = -t^{-2}\ell_{r-1}^2 = -t^{-2}\ell_r\end{aligned}$$

and so $\mathrm{lt}(g(M_r)) = 2\left(t^{-2}\ell_r\right) - \left(t^{-2}\ell_r\right) = t^{-2}\ell_r$, as desired. □

5. On the Jones polynomial of K_r

The *writhe* of an oriented link diagram D, denoted $\mathrm{wr}(D)$, is the sum of the signs of the crossings of D following conventions **a** and **b** of Figure 3.

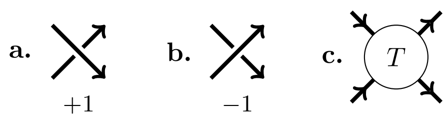


Figure 3. **a.** Crossing with positive writhe. **b.** Crossing with negative writhe. **c.** Left-right oriented tangle.

The notion of writhe can be naturally extended to an oriented tangle T. We say that a tangle T is *left-right orientable* if it can be equipped with an orientation as in **c.** of Figure 3, in which case we denote by $\mathrm{wr}(T)$ the writhe of T with respect to that orientation.

Lemma 5.1. *The tangle M_r is left-right orientable and we have* $\mathrm{wr}(M_r) = 0$.

Proof. We first remark that the tangle T_{10} is left-right orientable. Since $-T_{10}$ is obtained from T_{10} by switching the signs of the crossings, the tangle $-T_{10}$ is also left-right orientable and $\mathrm{wr}(-T_{10}) = -\mathrm{wr}(T_{10})$ holds. As the horizontal sum of tangles is compatible with the left-right orientation, for any left-right orientable tangles U and V, we have $\mathrm{wr}(U + V) = \mathrm{wr}(U) + \mathrm{wr}(V)$. As $M_1 = T_{10} + (-T_{10})$, we have

$$\mathrm{wr}(M_1) = \mathrm{wr}(T_{10}) + \mathrm{wr}(-T_{10}) = \mathrm{wr}(T_{10}) - \mathrm{wr}(T_{10}) = 0.$$

Again by the compatibility between the left-right orientation and the horizontal sum $+$, a straightforward induction yields $\mathrm{wr}(M_r) = \mathrm{wr}(M_{r-1}) + \mathrm{wr}(M_{r-1}) = 0 + 0 = 0$. □

The *normalized Kauffman bracket polynomial* of a link L depicted by an oriented diagram D is $\chi(L) = (-t^3)^{-\mathrm{wr}(D)}\langle D\rangle$, which is an invariant of the link L and so is independent of the choice of the oriented diagram D representing L. The *Jones polynomial* of a link L is then $V(L) = \chi(L)|_{t\leftarrow t^{-1/4}}$. The Kauffman bracket of the unknot $\bigcirc$ is $\langle \mathrm{den}(0)\rangle = 1$, which gives $\chi(\bigcirc) = 1$ and so $V(\bigcirc) = 1$.

Recall that K_r is the knot represented by the diagram $\mathrm{den}(1 * M_r)$.

Proposition 5.2. *For all $r \geq 1$, the Jones polynomial of K_r is equal to* 1 *modulo* 2^r*. Moreover the leading term of $\chi(K_r)$ is equal to $\ell_r = \left(2\, t^{28}\right)^{2^{r-1}}$.*

Proof. Let r be an integer ≥ 1. We denote by D_r the diagram $\mathrm{den}(1 * M_r)$. The left-right orientation of M_r induces the following orientation on D_r:

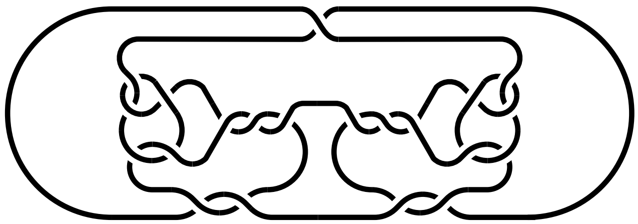
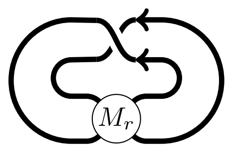


FIGURE 4. A mutant version K_1' of the knot K_1 obtained by replacing the tangle $-T_{10}$ in M_1 by its image under vertical symmetry. We can do the same process for all the knots K_r to obtain a mutant K_r'. As the Jones polynomial of a knot and of its mutants are the same [4], the Jones polynomial of K_r' is also trivial modulo 2^r for $r \geq 1$.



As the writhe of M_r is 0 by Lemma 5.1, the writhe of D_r with respect to the above orientation is +1.

Let us now determine the Kauffman bracket of D_r. We have

$$\operatorname{br}(1 * M_r) = \begin{bmatrix} \delta\, t\, f(M_r) + t\, g(M_r) + t^{-1}\, f(M_r) \\ t^{-1}\, g(M_r) \end{bmatrix}$$

and so

$$\begin{aligned} \langle D_r \rangle &= \delta\, t\, f(M_r) + t\, g(M_r) + t^{-1}\, f(M_r) + \delta\, t^{-1}\, g(M_r) \\ &= (\delta\, t + t^{-1})\, f(M_r) + (t + \delta\, t^{-1})\, g(M_r) \\ &= -t^3\, f(M_r) - t^{-3}\, g(M_r). \end{aligned}$$

Hence the normalized Kauffman bracket of K_r is

$$\begin{aligned} \chi(K_r) &= (-t^3)^{-\operatorname{wr}(D_r)} \cdot \langle D_R \rangle = (-t^3)^{-1}\, (-t^3\, f(M_r) - t^{-3}\, g(M_r)) \\ &= f(M_r) + t^{-6} g(M_r). \end{aligned}$$

As $f(M_r) = 1$ and $g(M_r) = 0$ modulo 2^r by Lemma 4.2, we obtain $\chi(K_r) = 1$ modulo 2^r and so $V(K_r)$ is trivial modulo 2^r.

By Lemma 4.2, the leading term of $f(M_r)$ is ℓ_r while that of $g(M_r)$ is $t^{-2}\ell_r$. Therefore the leading term of $\chi(K_r)$ is ℓ_r. □

We can now state and prove our main result.

Theorem 5.3. *For all $r \geq 1$, there exist infinitely many knots with trivial Jones polynomial modulo 2^r, namely the knots K_i for $i \geq r$.*

Proof. Let $r \geq 1$ be an integer. By Proposition 5.2, for all $i \geq r$ the Jones polynomial of K_i is trivial modulo 2^i and thus modulo 2^r. Since the leading term of $\chi(K_i)$ is ℓ_i for $i \geq 1$, the map $i \mapsto \chi(K_i)$ is injective. In particular the knots K_i have distinct Jones polynomials and so they are pairwise distinct. □

6. Concluding remarks

We have exhibited a 20-crossing prime tangle T_{20} whose Kauffman bracket polynomial pair is trivial mod 2, i.e. congruent to $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ mod 2. That tangle allowed us to construct, for any $r \geq 1$, a nontrivial knot K_r whose Jones polynomial is congruent to 1 mod 2^r.

Having thus solved Problem 1.1 for $m = 2^r$ and knowing solutions for $m = 3$ from the tables, what about the existence of solutions for the next moduli, such as $m = 5$, 6 or 7 for instance? More ambitiously perhaps, given an integer $m \geq 3$, does there exist a prime tangle, analogous to T_{20}, whose Kauffman bracket polynomial pair would be trivial mod m? If yes, what should the expected minimal number of crossings as a function of m be?

Even more intriguing: does there exist a prime tangle whose Kauffman bracket polynomial pair is trivial *over* $\mathbb{Z}$? The existence of such a tangle would probably imply the existence of a nontrivial knot with trivial Jones polynomial.

Acknowledgments. The authors wish to thank L. H. Kauffman, D. Rolfsen and M. B. Thistlethwaite for interesting discussions related to this work.

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